2.2 Index Notation for Vector and Tensor Operations

Operations on Cartesian components of vectors and tensors may be expressed very efficiently and clearly using *index notation***.**

2.1. Vector and tensor components.

Let **x** be a (three dimensional) vector and let **S** be a second order tensor. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Denote the components of **x** in this basis by (x_1, x_2, x_3), and denote the components of **S** by

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Using index notation, we would express **x** and **S** as

$$\mathbf{x} \equiv x_i \qquad \mathbf{S} \equiv S_{ij}$$

2.2. Conventions and special symbols for index notation

- **Range Convention:** Lower case Latin subscripts (*i, j, k...)* have the range $(1, 2, 3)$. The symbol x_i denotes three components of a vector $x_1,\ x_2$ and x_3. The symbol S_{ij} denotes nine components of a second order tensor, $S_{11},\ \ S_{12},\ \ S_{13},\ \ S_{21}\ \ldots\ S_{33}$

- **Summation convention (Einstein convention)**: If an index is repeated in a product of vectors or tensors, summation is implied over the repeated index. Thus

$$\lambda = a_i b_i \quad \equiv \quad \lambda = \sum_{i=1}^{3} a_i b_i \quad \equiv \quad \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$c_i = S_{ik} x_k \quad \equiv \quad c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \quad \begin{cases} c_1 = S_{11}x_1 + S_{12}x_2 + S_{13}x_3 \\ c_2 = S_{21}x_1 + S_{22}x_2 + S_{23}x_3 \\ c_3 = S_{31}x_1 + S_{32}x_2 + S_{33}x_3 \end{cases}$$

$$\lambda = S_{ij} S_{ij} \quad \equiv \quad \lambda = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \quad \equiv \quad \lambda = S_{11}S_{11} + S_{12}S_{12} + \ldots + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33}$$

$$C_{ij} = A_{ik} B_{kj} \quad \equiv \quad C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \quad \equiv \quad [C] = [A]\,[B]$$

$$C_{ij} = A_{ki} B_{kj} \quad \equiv \quad C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \quad \equiv \quad [C] = [A]^T\,[B]$$

In the last two equations, $[A]$, $[B]$ and $[C]$ denote the (3×3) component matrices of **A, B** and **C**.

- **The Kronecker Delta**: The symbol δ_{ij} is known as the Kronecker delta, and has the properties

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

thus

$$\delta_{11} = \delta_{22} = \delta_{33} = 1 \qquad \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{32} = \delta_{32} = 0$$

You can also think of δ_{ij} as the components of the identity tensor, or a (3×3) identity matrix. Observe the following useful results

$$\delta_{ij} = \delta_{ji}$$
$$\delta_{kk} = 3$$
$$a_i = \delta_{ik} a_k$$
$$A_{ij} = \delta_{ik} A_{kj}$$

● **The Permutation Symbol:** The symbol $\in_{ijk}$ has properties

$$\in_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3; \quad 2,3,1 \quad \text{or} \quad 3,1,2 \\ -1 & i,j,k = \ 3,2,1; \quad 2,1,3 \quad \text{or} \quad 1,3,2 \\ 0 & \text{otherwise} \end{cases}$$

thus

$$\in_{123} = \in_{231} = \in_{312} = 1$$
$$\in_{321} = \in_{213} = \in_{132} = -1$$
$$\in_{111} = \in_{112} = \in_{113} = \in_{121} = \in_{122} = \in_{131} = \in_{133} = 0$$
$$\in_{211} = \in_{212} = \in_{221} = \in_{222} = \in_{223} = \in_{232} = \in_{233} = 0$$
$$\in_{311} = \in_{313} = \in_{322} = \in_{323} = \in_{321} = \in_{332} = \in_{333} = 0$$

Note that

$$\in_{ijk} = \in_{kij} = \in_{jki} = -\in_{jik} = -\in_{kji} = -\in_{kji}$$
$$\in_{kki} = 0$$
$$\in_{ijk} \in_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$$
$$\in_{ijk} \in_{lmn} = \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)$$

2.3. Rules of index notation

1. The same index (subscript) may not appear more than twice in a product of two (or more) vectors or tensors. Thus

$$A_{ik}x_k, \quad A_{ik}B_{kj}, \quad A_{ij}B_{ik}C_{nk}$$

are valid, but

$$A_{kk}x_k, \quad A_{ik}B_{kk}, \quad A_{ij}B_{ik}C_{ik}$$

are meaningless

2. Free indices on each term of an equation must agree. Thus

$$x_i = u_i + c_i \qquad \equiv \qquad \mathbf{x} = \mathbf{u} + \mathbf{c}$$
$$a_i = A_{ki}B_{kj}x_j + C_{ik}u_k \qquad \equiv \qquad \mathbf{a} = \mathbf{A}^T\mathbf{B}\,\mathbf{x} + \mathbf{C}\mathbf{u}$$

are valid, but

$$x_i = A_{ij}$$
$$x_j = A_{ik}u_k$$
$$x_i = A_{ik}u_k + c_j$$

are meaningless.

3. Free and dummy indices may be changed without altering the meaning of an expression, provided that rules 1 and 2 are not violated. Thus

$$x_i = A_{ik}x_k \Leftrightarrow x_j = A_{jk}x_k \Leftrightarrow x_j = A_{ji}x_i$$

2.4. Vector operations expressed using index notation

● **Addition.** $\mathbf{c} = \mathbf{a} + \mathbf{b} \quad \equiv \quad c_i = a_i + b_i$

● **Dot Product** $\lambda = \mathbf{a} \cdot \mathbf{b} \quad \equiv \quad \lambda = a_i b_i$

● **Vector Product** $\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \equiv \quad c_i = \in_{ijk} a_j b_k$

- **Dyadic Product** $\mathbf{S}=\mathbf{a}\otimes\mathbf{b} \quad \equiv \quad S_{ij}=a_i b_j$

- **Change of Basis**. Let **a** be a vector. Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis, and denote the components of **a** in this basis by a_i. Let $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ be a second basis, and denote the components of **a** in this basis by α_i. Then, define
$$Q_{ij}=\mathbf{m}_i\cdot\mathbf{e}_j=\cos\theta(\mathbf{m}_i,\mathbf{e}_j)$$
where $\theta(\mathbf{m}_i,\mathbf{e}_j)$ denotes the angle between the unit vectors $\mathbf{m}_i$ and $\mathbf{e}_j$. Then
$$\alpha_i=Q_{ij}a_j$$

2.5. Tensor operations expressed using index notation.

- **Addition.** $\mathbf{C}=\mathbf{A}+\mathbf{B} \quad \equiv \quad C_{ij}=A_{ij}+B_{ij}$

- **Transpose** $\mathbf{A}=\mathbf{B}^T \quad \equiv \quad A_{ij}=B_{ji}$

- **Scalar Products** $\lambda=\mathbf{A}:\mathbf{B} \quad \equiv \quad \lambda=A_{ij}B_{ij}$
  $\lambda=\mathbf{A}\cdot\cdot\mathbf{B} \quad \equiv \lambda=A_{ji}B_{ij}$

- **Product of a tensor and a vector** $\mathbf{c}=\mathbf{A}\mathbf{b} \quad \equiv \quad c_i=A_{ij}b_j$
  $\mathbf{c}=\mathbf{A}^T\mathbf{b} \quad \equiv \quad c_i=A_{ji}b_j$

- **Product of two tensors** $\mathbf{C}=\mathbf{A}\mathbf{B} \quad \equiv \quad C_{ij}=A_{ik}B_{kj}$
  $\mathbf{C}=\mathbf{A}^T\mathbf{B} \quad \equiv \quad C_{ij}=A_{ki}B_{kj}$

- **Determinant** $\lambda=\det\mathbf{A} \quad \equiv \quad \lambda=\frac{1}{6}\in_{ijk}\in_{lmn} A_{li}A_{mj}A_{nk}=\in_{ijk} A_{i1}A_{j2}A_{k3}$
  $\Leftrightarrow \quad \in_{lmn}\lambda = \quad \in_{ijk} A_{li}A_{mj}A_{nk}=\in_{ijk} A_{il}A_{jm}A_{kn}$

- **Inverse** $S^{-1}_{ji}=\frac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl} S_{pk}S_{ql}$

- **Change of Basis**. Let **A** be a second order tensor. Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis, and denote the components of **A** in this basis by A_{ij}. Let $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ be a second basis, and denote the components of **A** in this basis by Λ_{ij}. Then, define
$$Q_{ij}=\mathbf{m}_i\cdot\mathbf{e}_j=\cos\theta(\mathbf{m}_i,\mathbf{e}_j)$$
where $\theta(\mathbf{m}_i,\mathbf{e}_j)$ denotes the angle between the unit vectors $\mathbf{m}_i$ and $\mathbf{e}_j$. Then
$$\Lambda_{ij}=Q_{ik}A_{km}Q_{jm}$$

2.6. Calculus using index notation

The derivative $\partial x_i/\partial x_j$ can be deduced by noting that $\partial x_i/\partial x_j=1 \;\; i=j$ and $\partial x_i/\partial x_j=0 \;\; i\neq j$. Therefore
$$\frac{\partial x_i}{\partial x_j}=\delta_{ij}$$
The same argument can be used for higher order tensors
$$\frac{\partial A_{ij}}{\partial A_{kl}}=\delta_{ik}\delta_{jl}$$

2.7. Examples of algebraic manipulations using index notation

1. Let **a, b, c, d** be vectors. Prove that
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$

Express the left hand side of the equation using index notation (check the rules for cross products and dot products of vectors to see how this is done)
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \quad \equiv \quad \in_{ijk} a_j b_k \in_{imn} c_m d_n$$
Recall the identity
$$\in_{ijk}\in_{imn}=\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{mk}$$
so
$$\in_{ijk} a_j b_k \in_{imn} c_m d_n=(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{mk})\,a_j b_k c_m d_n$$
Multiply out, and note that
$$\delta_{jm}a_j=a_m \qquad \delta_{kn}b_k=b_n$$
(multiplying by a Kronecker delta has the effect of switching indices…) so
$$(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{mk})\,a_j b_k c_m d_n=a_m b_n c_m d_n-a_n b_m c_m d_n$$
Finally, note that
$$a_m c_m \quad \equiv \mathbf{a}\cdot\mathbf{c}$$
and similarly for other products with the same index, so that
$$a_m b_n c_m d_n-a_n b_m c_m d_n=a_m c_m b_n d_n-b_m c_m a_n d_n\equiv(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$

2. The stress—strain relation for linear elasticity may be expressed as

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

where σ_{ij} and ε_{ij} are the components of the stress and strain tensor, and E and ν denote Young's modulus and Poisson's ratio. Find an expression for strain in terms of stress.

Set *i=j* to see that

$$\sigma_{ii} = \frac{E}{1+\nu}\left(\varepsilon_{ii} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ii}\right)$$

Recall that $\delta_{ii} = 3$, and notice that we can replace the remaining *ii* by *kk*

$$\sigma_{kk} = \frac{E}{1+\nu}\left(\varepsilon_{kk} + \frac{\nu}{1-2\nu}3\varepsilon_{kk}\right) = \frac{E}{1-2\nu}\varepsilon_{kk}$$

$$\Leftrightarrow \quad \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$$

Now, substitute for ε_{kk} in the given stress—strain relation

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)$$

$$\Leftrightarrow \varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right)$$

3. Solve the equation

$$\mu\left\{\delta_{kj}a_i a_i + \frac{1}{1-2\nu}a_k a_j\right\}U_k = P_j$$

for U_k in terms of P_i and a_i

Multiply both sides by a_j to see that

$$\mu\left\{a_j\delta_{kj}a_i a_i + \frac{1}{1-2\nu}a_k a_j a_j\right\}U_k = P_j a_j$$

$$\Leftrightarrow \quad \mu\left\{a_k a_i a_i + \frac{1}{1-2\nu}a_k a_j a_j\right\}U_k = P_j a_j$$

$$\Leftrightarrow \mu U_k a_k \frac{2(1-\nu)}{1-2\nu}a_i a_i = P_j a_j \quad \Leftrightarrow U_k a_k = \frac{(1-2\nu)P_j a_j}{2\mu(1-\nu)a_i a_i}$$

Substitute back into the equation given for $U_k a_k$ to see that

$$\mu U_j a_i a_i + \frac{P_k a_k}{2(1-\nu)a_i a_i}a_j = P_j \quad \Rightarrow U_j = \frac{1}{\mu a_i a_i}\left(P_j - \frac{P_k a_k}{2(1-\nu)a_n a_n}a_j\right)$$

4. Let $r = \sqrt{x_k x_k}$. Calculate $\frac{\partial r}{\partial x_i}$

We can just apply the usual chain and product rules of differentiation

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\frac{1}{\sqrt{x_k x_k}}\left(x_k\frac{\partial x_k}{\partial x_i} + \frac{\partial x_k}{\partial x_i}x_k\right) = \frac{1}{\sqrt{x_k x_k}}x_k\delta_{ik} = \frac{x_i}{\sqrt{x_k x_k}} = \frac{x_i}{r}$$

5. Let $\lambda = A_{ij}A_{ij}$. Calculate $\partial\lambda/\partial A_{kl}$

Using the product rule

$$\frac{\partial\lambda}{\partial A_{kl}} = A_{ij}\delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jl}A_{ij} = 2A_{kl}$$